

A WEAK GALERKIN FINITE ELEMENT SCHEME FOR SOLVING THE STATIONARY STOKES EQUATIONS

RUI SHU WANG*, XIAOSHEN WANG†, QILONG ZHAI ‡, AND RAN ZHANG§

Abstract. A weak Galerkin (WG) finite element method for solving the stationary Stokes equations in two- or three- dimensional spaces by using discontinuous piecewise polynomials is developed and analyzed. The variational form we considered is based on two gradient operators which is different from the usual gradient-divergence operators. The WG method is highly flexible by allowing the use of discontinuous functions on arbitrary polygons or polyhedra with certain shape regularity. Optimal-order error estimates are established for the corresponding WG finite element solutions in various norms. Numerical results are presented to illustrate the theoretical analysis of the new WG finite element scheme for Stokes problems.

Key words. weak Galerkin finite element methods, weak gradient, Stokes equations, polytopal meshes.

AMS subject classifications. Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45, 35J50, 35J35

1. Introduction. The aim of this paper is to present a novel weak Galerkin finite element method for solving the stationary Stokes equations. Let Ω be a polygonal or polyhedral domain in $\mathbb{R}^d, d = 2, 3$. As a model for the flow of an incompressible viscous fluid confined in Ω , we consider the following equations

$$(1.1) \quad -\mu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega,$$

$$(1.2) \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega,$$

$$(1.3) \quad \mathbf{u} = \mathbf{g}, \quad \text{on } \partial\Omega,$$

for unknown velocity function \mathbf{u} and pressure function p (we require that p has zero average in order to guarantee the uniqueness of the pressure). Bold symbols are used to denote vector- or tensor-valued functions or spaces of such functions. Here \mathbf{f} is a body source term, $\mu > 0$ is the kinematic viscosity and \mathbf{g} is a boundary condition that satisfies the compatibility condition

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, ds = 0,$$

where \mathbf{n} is the unit outward normal vector on the domain boundary $\partial\Omega$.

This problem mainly arises from approximations of low-Reynolds-number flows. The finite element methods for Stokes and Navier–Stokes problems enforce the divergence-free property in finite element spaces, which satisfy the inf-sup (LBB)

*Department of Mathematics, Jilin University, Changchun, China (ruishu@email.jlu.edu.cn).

†Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, United States (xxwang@ualr.edu).

‡Department of Mathematics, Jilin University, Changchun, China (diql13@mails.jlu.edu.cn).

§Department of Mathematics, Jilin University, Changchun, China (zhangran@mail.jlu.edu.cn). The research of Zhang was supported in part by China Natural National Science Foundation (11271157, 11371171, 11471141), and by the Program for New Century Excellent Talents in University of Ministry of Education of China.

condition, in order for them to be numerically stable [2, 1, 10, 11, 8]. The Stokes problem has been studied with various different new numerical methods: [4, 12, 13, 22, 23].

Throughout this paper, we would follow the standard definitions for Lebesgue and Sobolev spaces: $L^2(\Omega)$, $H^1(\Omega)$, $[L^2(\Omega)]^d$,

$$[H_0^1(\Omega)]^d = \{\mathbf{v} \in [H^1(\Omega)]^d : \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\}$$

and

$$L_0^2(\Omega) := \{q \in L^2(\Omega) : \int_{\Omega} q dx = 0\}$$

are the natural spaces for the weak form of the Stokes problem [10, 7]. Denote (\cdot, \cdot) for inner products in the corresponding spaces.

Next we assume that $\mu = 1$ and $\mathbf{g} = \mathbf{0}$. Then one of the variational formulations for the Stokes problem (1.1)-(1.3) is to find $\mathbf{u} \in [H_0^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$ such that

$$(1.4) \quad (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) = (\mathbf{f}, \mathbf{v}),$$

$$(1.5) \quad (\nabla \cdot \mathbf{u}, q) = 0,$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$. Here $\nabla \mathbf{u}$ denotes the velocity gradient tensor $(\nabla \mathbf{u})_{ij} = \partial_j \mathbf{u}_i$. It is well known that under our assumptions on the domain and the data, problem (1.4)-(1.5) has a unique solution $(\mathbf{u}; p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$.

For any $p \in L_0^2(\Omega)$, define a functional ∇p such that

$$\langle \nabla p, \mathbf{v} \rangle = -(\nabla \cdot \mathbf{v}, p), \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^d.$$

It is easy to know that the weak form (1.4)-(1.5) is also equivalent to the following variational problem: find $(\mathbf{u}; p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ such that

$$(1.6) \quad (\nabla \mathbf{u}, \nabla \mathbf{v}) + \langle \nabla p, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v}),$$

$$(1.7) \quad \langle \nabla q, \mathbf{u} \rangle = 0,$$

for all $\mathbf{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$. The unique solvability of (1.6)-(1.7) follows directly from that of the (1.4)-(1.5).

The WG method refers to a general finite element technique for partial differential equations where differential operators are approximated as distributions for generalized functions. This method was first proposed in [20, 21, 15] for second order elliptic problem, then extended to other partial differential equations [14, 16, 18, 17, 25, 26]. Weak functions and weak derivatives can be approximated by polynomials with various degrees. The WG method uses weak functions and their weak derivatives which are defined as distributions. The most prominent features of it are:

- The usual derivatives are replaced by distributions or discrete approximations of distributions.
- The approximating functions are discontinuous. The flexibility of discontinuous functions gives WG methods many advantages, such as high order of accuracy, high parallelizability, localizability, and easy handling of complicated geometries.

The above features motivate the use of WG methods for the Stokes equations. It can easily handle meshes with hanging nodes, elements of general shapes with certain shape regularity and ideally suited for hp-adaptivity. In [19], Wang et. al. considered WG methods for the Stokes equations (1.4)-(1.5). Similarly, in [17], they presented WG methods for the Brinkman equations, which is a model with a high-contrast parameter dependent combination of the Darcy and Stokes models. The numerical method of [17] is based on the traditional gradient-divergence variational form for the Brinkman equations. In [24], we presented a new WG scheme based on the gradient-gradient variational form. It is shown that this scheme is suit for the mixed formulation of Darcy which would present a better approximation for this case. In fact, for complex porous media with interface conditions, people often use Brinkman-Stokes interface model to describe this problem, which is an ongoing work for us now. In order to present a more efficient WG scheme, we prefer to utilize this gradient-gradient weak form to approximate the model. In order to unify the weak form of this interface problem, we need the numerical analysis results of this form for Stokes problem. However, to the best of our knowledge, the numerical analysis of methods based on the variational form (1.6)-(1.7) has never been done before. Therefore in this paper, we propose a WG method based on the weak form (1.6)-(1.7) of the primary problem. In addition, if we choose high order polynomials to approximate the model and use Schur complement to reduce the interior DOF of the velocity and pressure by the boundary DOF, the total DOF of this new method is less than the scheme of [19].

The rest of this paper is organized as follows. In Section 2 we shall introduce some preliminaries and notations for Sobolev spaces. Section 3 is devoted to the definitions of weak functions and weak derivatives. The WG finite element schemes for variational form of the Stokes equation (1.6)-(1.7) are presented in Section 4. This section also contains some local L^2 projection operators and then derives some approximation properties which are useful in a convergence analysis. In Section 5, we derive an error equation for the WG finite element approximation. Optimal-order error estimates for the WG finite element approximations are derived in Section 6 in an H^1 -equivalent norm for the velocity, and L^2 norm for both the velocity and the pressure. In Section 7, we present some numerical results which confirm the theory developed in earlier sections. Finally, we present some technical estimates in the appendix for quantities related to the local L^2 projections into various finite element spaces.

2. Preliminaries and Notations. Let $K \subset \Omega$ be an open bounded domain with Lipschitz continuous boundary in \mathbb{R}^d , $d = 2, 3$. We shall use standard definitions of the Sobolev spaces $H^s(K)$ and inner products $(\cdot, \cdot)_{s,K}$, their norms $\|\cdot\|_{s,K}$, and seminorms $|\cdot|_{s,K}$, for any $s \geq 0$. For instance, for any integer $s \geq 0$, the seminorm $|\cdot|_{s,K}$ is defined as

$$|v|_{s,K} = \left(\sum_{|\alpha|=s} \int_K |\partial^\alpha v|^2 dK \right)^{\frac{1}{2}},$$

with notations

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d), \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d, \quad \partial^\alpha = \prod_{j=1}^d \partial_{x_j}^{\alpha_j}.$$

The Sobolev norm $\|\cdot\|_{m,K}$ is defined as

$$\|v\|_{m,K} = \left(\sum_{j=0}^m |v|_{j,K}^2 \right)^{\frac{1}{2}}.$$

The space $H^0(K)$ is same as $L^2(K)$, whose norm and inner product are denoted by $\|\cdot\|_K$ and $(\cdot, \cdot)_K$, respectively. If $K = \Omega$, we would drop the subscript K in the notations of the L^2 norm and the L^2 inner product.

3. Weak Differential Operators. In this section we will define weak functions for both the vector-valued function and the scalar-valued function, also we will introduce the weak gradients and the corresponding discrete forms.

3.1. Weak gradient for weak vector-valued function. Let T be a polygonal or polyhedral domain with boundary ∂T . A weak vector-valued function on the domain T is defined as $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$ such that $\mathbf{v}_0 \in [L^2(T)]^d$ and $\mathbf{v}_b \in [L^2(\partial T)]^d$. Let

$$V(T) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}; \mathbf{v}_0 \in [L^2(T)]^d, \mathbf{v}_b \in [L^2(\partial T)]^d\},$$

where \mathbf{v}_b is not necessarily the trace of \mathbf{v}_0 .

DEFINITION 3.1. ([19]) *For any $\mathbf{v} \in V(T)$, the weak gradient of \mathbf{v} , denoted by $\nabla_w \mathbf{v}$, is defined as a linear functional in the dual space of $[H^1(T)]^{d \times d}$ whose action on each $\tau \in [H^1(T)]^{d \times d}$ is given by*

$$(\nabla_w \mathbf{v}, \tau)_T = -(\mathbf{v}_0, \nabla \cdot \tau)_T + \langle \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \tau \in [H^1(T)]^{d \times d},$$

where \mathbf{n} is the outer unit normal vector to ∂T , $(\mathbf{v}_0, \nabla \cdot \tau)_T$ is the L^2 inner product of \mathbf{v}_0 and $\nabla \cdot \tau$, and $\langle \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T}$ is the inner product of $\tau \cdot \mathbf{n}$ and \mathbf{v}_b in $[L^2(\partial T)]^d$.

Consider the inclusion map $i_V : [H^1(T)]^d \rightarrow V(T)$ defined below

$$i_V(\phi) = \{\phi|_T, \phi|_{\partial T}\}, \quad \phi \in [H^1(T)]^d.$$

By this map the Sobolev space $[H^1(T)]^d$ can be embedded into the space $V(T)$. With the help of map i_V , the Sobolev space $[H^1(T)]^d$ can be considered as a subspace of $V(T)$ by identifying each $\phi \in [H^1(T)]^d$ with $i_V(\phi)$.

Let $P_r(T)$ be the set of polynomials on T with degree no more than r .

DEFINITION 3.2. ([19]) *The discrete weak gradient operator $\nabla_{w,r,T}$ is defined as follows: for each $\mathbf{v} \in V(T)$, $\nabla_{w,r,T} \mathbf{v} \in [P_r(T)]^{d \times d}$ is the unique element such that*

$$(3.1) \quad (\nabla_{w,r,T} \mathbf{v}, \tau)_T = -(\mathbf{v}_0, \nabla \cdot \tau)_T + \langle \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \tau \in [P_r(T)]^{d \times d}.$$

3.2. Weak gradient for weak scalar-valued function. We define a weak scalar-valued function on the domain T as $q = \{q_0, q_b\}$ such that $q_0 \in L^2(T)$ and $q_b \in L^2(\partial T)$. Let

$$W(T) = \{q = \{q_0, q_b\}; q_0 \in L^2(T), q_b \in L^2(\partial T)\},$$

where q_b is not necessarily the trace of q_0 .

DEFINITION 3.3. ([20]) For any $q \in W(T)$, the weak gradient of q , denote by $\tilde{\nabla}_w q$, is defined as a linear functional in the dual space of $[H^1(T)]^2$ whose action on each $\mathbf{w} \in [H^1(T)]^2$ is given by

$$(\tilde{\nabla}_w q, \mathbf{w})_T = -(q_0, \nabla \cdot \mathbf{w})_T + \langle q_b, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{w} \in [H^1(T)]^2,$$

where \mathbf{n} is the outer unit normal vector to ∂T , $(q_0, \nabla \cdot \mathbf{w})_T$ is the L^2 inner product of q_0 and $\nabla \cdot \mathbf{w}$, and $\langle q_b, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T}$ is the inner product of $\mathbf{w} \cdot \mathbf{n}$ and q_b in $L^2(\partial T)$.

Consider the inclusion map $i_W : H^1(T) \rightarrow W(T)$ defined as follows

$$i_W(\phi) = \{\phi|_T, \phi|_{\partial T}\}, \quad \phi \in H^1(T).$$

By which the Sobolev space $H^1(T)$ is embedded into the space $W(T)$. With the help of map i_W , the Sobolev space $H^1(T)$ can be considered as a subspace of $W(T)$ by identifying each $\phi \in H^1(T)$ with $i_W(\phi)$.

DEFINITION 3.4. ([20]) The discrete weak gradient operator $\tilde{\nabla}_{w,r,T}$ is defined as follows: for each $q \in W(T)$, $\tilde{\nabla}_{w,r,T} q \in [P_r(T)]^d$ is the unique element such that

$$(3.2) \quad (\tilde{\nabla}_{w,r,T} q, \mathbf{w})_T = -(q_0, \nabla \cdot \mathbf{w})_T + \langle q_b, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{w} \in [P_r(T)]^d.$$

4. A Weak Galerkin Finite Element Scheme. Let \mathcal{T}_h be a partition of the domain Ω into polygons in 2D or polyhedral in 3D. Assume that \mathcal{T}_h is shape regular in the sense as defined in [18]. Denote by \mathcal{E}_h the set of all edges or flat faces in \mathcal{T}_h , and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ be the set of all interior edges or flat faces. Denote by h_T the diameter of $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize for the partition \mathcal{T}_h .

For any interger $k \geq 1$, we define weak Galerkin finite element spaces as follows: for velocity variable, let

$$V_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}; \{\mathbf{v}_0, \mathbf{v}_b\}|_T \in [P_k(T)]^d \times [P_k(e)]^d, e \subset \partial T, \mathbf{v}_b = \mathbf{0} \text{ on } \partial\Omega\}.$$

It should be noticed that \mathbf{v}_b is single valued on each edge $e \subset \mathcal{E}_h$. For pressure variable, we define

$$W_h = \left\{ q = \{q_0, q_b\}; \sum_{T \in \mathcal{T}_h} \int_T q_0 dT = 0, \{q_0, q_b\}|_T \in P_{k-1}(T) \times P_k(e), e \subset \partial T \right\}.$$

Also q_b is single valued on each edge $e \subset \mathcal{E}_h$.

The discrete weak gradients $\nabla_{w,k-1}$ and $\tilde{\nabla}_{w,k}$ on the spaces V_h and W_h can be computed by the equations (3.1) and (3.2) on each element T respectively, that is,

$$\begin{aligned} (\nabla_{w,k-1} \mathbf{v})|_T &= \nabla_{w,k-1,T}(\mathbf{v}|_T), \quad \forall \mathbf{v} \in V_h, \\ (\tilde{\nabla}_{w,k} q)|_T &= \tilde{\nabla}_{w,k,T}(q|_T), \quad \forall q \in W_h. \end{aligned}$$

For the sake of simplicity, we shall drop the subscripts $k-1$ and k of $\nabla_{w,k-1}$ and $\tilde{\nabla}_{w,k}$ in the rest of the paper.

We use the L^2 inner product to denote the sum of inner products on each of the elements as follows:

$$\begin{aligned}(\nabla_w \mathbf{v}, \nabla_w \mathbf{w}) &= \sum_{T \in \mathcal{T}_h} (\nabla_w \mathbf{v}, \nabla_w \mathbf{w})_T, \\(\tilde{\nabla}_w q, \mathbf{v}) &= \sum_{T \in \mathcal{T}_h} (\tilde{\nabla}_w q, \mathbf{v})_T.\end{aligned}$$

LEMMA 4.1. ([19]) *For any $\mathbf{v} \in V_h$ and $p \in W_h$ the following equations hold true*

$$(4.1) \quad (\nabla_w \mathbf{v}, \tau)_T = (\nabla \mathbf{v}_0, \tau)_T - \langle \mathbf{v}_0 - \mathbf{v}_b, \tau \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \tau \in [P_{k-1}(T)]^{d \times d},$$

$$(4.2) \quad (\tilde{\nabla}_w p, \mathbf{w})_T = (\nabla p_0, \mathbf{w})_T - \langle p_0 - p_b, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{w} \in [P_k(T)]^d.$$

For each element $T \in \mathcal{T}_h$, denote by Q_0 the L^2 projection operator from $[L^2(T)]^d$ onto $[P_k(T)]^d$. For each edge or face $e \in \mathcal{E}_h$, denote by Q_b the L^2 projection from $[L^2(e)]^d$ onto $[P_k(e)]^d$. We shall combine Q_0 with Q_b as a projection onto V_h , such that on each element $T \in \mathcal{T}_h$

$$Q_h \mathbf{u} = \{Q_0 \mathbf{u}, Q_b \mathbf{u}\}.$$

On each element $T \in \mathcal{T}_h$, denote by \mathbf{Q}_h the L^2 projection onto $[P_{k-1}(T)]^{d \times d}$. Denote by \tilde{Q}_0 the L^2 projection operator from $L^2(T)$ onto $P_{k-1}(T)$. For each edge or face $e \in \mathcal{E}_h$, denote by \tilde{Q}_b the L^2 projection from $L^2(e)$ onto $P_k(e)$. We shall combine \tilde{Q}_0 with \tilde{Q}_b as a projection onto space W_h , such that on each element $T \in \mathcal{T}_h$

$$\tilde{Q}_h q = \{\tilde{Q}_0 q, \tilde{Q}_b q\}.$$

Then we shall present a useful property which indicates the discrete weak gradient operators are good approximation to the gradient operators in the classical sense.

LEMMA 4.2. ([19]) *The following equations hold true.*

$$(4.3) \quad \nabla_w Q_h \mathbf{v} = \mathbf{Q}_h \nabla \mathbf{v}, \quad \forall \mathbf{v} \in [H^1(\Omega)]^d,$$

$$(4.4) \quad \tilde{\nabla}_w \tilde{Q}_h p = Q_0 \nabla p, \quad \forall p \in H^1(\Omega).$$

Now we introduce four bilinear forms as follows:

$$(4.5) \quad s(\mathbf{w}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \mathbf{w}_0 - \mathbf{w}_b, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial T},$$

$$(4.6) \quad a(\mathbf{w}, \mathbf{v}) = (\nabla_w \mathbf{w}, \nabla_w \mathbf{v}) + s(\mathbf{w}, \mathbf{v}),$$

$$(4.7) \quad b(\mathbf{w}, q) = (\mathbf{w}_0, \tilde{\nabla}_w q),$$

$$(4.8) \quad c(\rho, q) = \sum_{T \in \mathcal{T}_h} h_T \langle \rho_0 - \rho_b, q_0 - q_b \rangle_{\partial T}.$$

Using these bilinear forms we define the following two norms. For any $\mathbf{v} \in V_h^0$ and $q \in W_h$,

$$(4.9) \quad \|\mathbf{v}\|^2 = a(\mathbf{v}, \mathbf{v}) = (\nabla_w \mathbf{v}, \nabla_w \mathbf{v}) + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle \mathbf{v}_0 - \mathbf{v}_b, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial T},$$

and

$$(4.10) \quad \|q\|_0^2 = \|q_0\|^2 + \|q\|_*^2,$$

where $\|q\|_*^2 = c(q, q)$ is a seminorm.

It is easy to verify that $\|\cdot\|$ and $\|\cdot\|_0$ are norms in V_h and W_h , respectively,

WEAK GALERKIN ALGORITHM 1. *A numerical approximation for (1.1)-(1.3) can be obtained by seeking $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_h$ and $p_h = \{p_0, p_b\} \in W_h$ such that*

$$(4.11) \quad a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}_0),$$

$$(4.12) \quad b(\mathbf{u}_h, q) - c(p_h, q) = 0,$$

for all $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$ and $q \in W_h$.

Next we shall show that the weak Galerkin finite element algorithm (4.11)-(4.12) has only one solution. Since the system is linear, it suffices to show that if $\mathbf{f} = \mathbf{0}$, the only solution is $\mathbf{u}_h = \{\mathbf{0}, \mathbf{0}\}; p_h = \{0, 0\}$.

LEMMA 4.3. *The WG finite element scheme (4.11)-(4.12) has a unique solution.*

Proof. Let $\mathbf{f} = \mathbf{0}$, we shall show that the solution of (4.11)-(4.12) is trivial. To this end, taking $\mathbf{v} = \mathbf{u}_h$ and $q = p_h$ and subtracting (4.12) from (4.11) we arrive at

$$a(\mathbf{u}_h, \mathbf{u}_h) + c(p_h, p_h) = 0.$$

By the definition of $a(\cdot, \cdot)$ and $c(\cdot, \cdot)$, we know $\nabla_w \mathbf{u}_h = 0$ on each $T \in \mathcal{T}_h$, $\mathbf{u}_0 = \mathbf{u}_b$, and $p_0 = p_b$ on each ∂T . Thus \mathbf{u}_0 and q_0 are continuous.

By (4.1) and the fact that $\mathbf{u}_b = \mathbf{u}_0$ on ∂T we have, for any $\tau \in [P_{k-1}(T)]^{d \times d}$,

$$\begin{aligned} 0 &= (\nabla_w \mathbf{u}_h, \tau)_T \\ &= (\nabla \mathbf{u}_0, \tau)_T - \langle \mathbf{u}_0 - \mathbf{u}_b, \tau \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla \mathbf{u}_0, \tau)_T, \end{aligned}$$

which implies $\nabla \mathbf{u}_0 = 0$ on each $T \in \mathcal{T}_h$ and thus \mathbf{u}_0 is a constant. Since $\mathbf{u}_0 = \mathbf{u}_b$ on each ∂T and $\mathbf{u}_b = \mathbf{0}$ on $\partial \Omega$, we arrive at $\mathbf{u}_h = \{\mathbf{0}, \mathbf{0}\}$ in Ω . It follows from (4.11), $\mathbf{u}_h = \{\mathbf{0}, \mathbf{0}\}$, and $\mathbf{f} = \mathbf{0}$ that for any $\mathbf{v} \in V_h$,

$$\begin{aligned} 0 &= b(\mathbf{v}, p_h) \\ &= (\mathbf{v}_0, \hat{\nabla}_w p_h) \\ &= \sum_{T \in \mathcal{T}_h} (\mathbf{v}_0, \nabla p_0)_T - \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_0 \cdot \mathbf{n}, p_0 - p_b \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (\mathbf{v}_0, \nabla p_0)_T. \end{aligned}$$

Hence we have $\nabla p_0 = 0$ on each $T \in \mathcal{T}_h$. Thus p_0 is a constant in Ω . From $p_0 \in L_0^2(\Omega)$, we would obtain $p_0 = 0$ in Ω . Since $p_b = p_0$ on each ∂T , $p_b = 0$.

This completes the proof of the lemma. \square

5. Error Equation. In this section, we shall derive the error equations for the WG finite element solution we get from (4.11)-(4.12). This error equation is essential for the following analysis.

Now we define two bilinear forms

$$(5.1) \quad l_1(\mathbf{w}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_0 - \mathbf{v}_b, (\nabla \mathbf{w} - \mathbf{Q}_h \nabla \mathbf{w}) \cdot \mathbf{n} \rangle_{\partial T},$$

$$(5.2) \quad l_2(\mathbf{w}, q) = \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, (\mathbf{w} - Q_0 \mathbf{w}) \cdot \mathbf{n} \rangle_{\partial T},$$

for all $\mathbf{w} \in [H^1(\Omega)]^d$, $\mathbf{v} \in V_h$ and $q \in W_h$.

Let $(\mathbf{u}; p)$ be the exact solution of (1.1)-(1.3), and $(\mathbf{u}_h; p_h) \in V_h \times W_h$ be the solution of (4.11)-(4.12).

Define

$$\mathbf{e}_h = Q_h \mathbf{u} - \mathbf{u}_h, \quad \varepsilon_h = \tilde{Q}_h p - p_h.$$

We shall derive the error equations that $\mathbf{e}_h \in V_h$ and $\varepsilon_h \in W_h$ satisfy.

LEMMA 5.1. *Let $\mathbf{u}_h \in V_h$ and $p_h \in W_h$ be the solution of the numerical scheme (4.11)-(4.12), and $(\mathbf{u}; p)$ be the exact solution of (1.1)-(1.3). Then, for any $\mathbf{v} \in V_h$ and $q \in W_h$ we have*

$$(5.3) \quad a(\mathbf{e}_h, \mathbf{v}) + b(\mathbf{v}, \varepsilon_h) = s(Q_h \mathbf{u}, \mathbf{v}) + l_1(\mathbf{u}, \mathbf{v}),$$

$$(5.4) \quad b(\mathbf{e}_h, q) - c(\varepsilon_h, q) = l_2(\mathbf{u}, q) - c(\tilde{Q}_h p, q).$$

Proof. First, from (4.1) and the property (4.3) we obtain

$$\begin{aligned} (\nabla_w Q_h \mathbf{u}, \nabla_w \mathbf{v})_T &= (\mathbf{Q}_h \nabla \mathbf{u}, \nabla_w \mathbf{v})_T \\ &= (\nabla \mathbf{v}_0, \mathbf{Q}_h \nabla \mathbf{u})_T - \langle \mathbf{v}_0 - \mathbf{v}_b, (\mathbf{Q}_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla \mathbf{v}_0, \nabla \mathbf{u})_T - \langle \mathbf{v}_0 - \mathbf{v}_b, (\mathbf{Q}_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T}. \end{aligned}$$

Summing over all elements $T \in \mathcal{T}_h$, we have

$$(5.5) \quad (\nabla_w Q_h \mathbf{u}, \nabla_w \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}_0) - \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_0 - \mathbf{v}_b, (\mathbf{Q}_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T}.$$

From the commutative property (4.4) we arrive at

$$(5.6) \quad b(\mathbf{v}, \tilde{Q}_h p) = (\mathbf{v}_0, \tilde{\nabla}_w \tilde{Q}_h p) = (\mathbf{v}_0, \nabla p).$$

It follows from (4.2) that

$$\begin{aligned} (Q_0 \mathbf{u}, \tilde{\nabla}_w q)_T &= (\nabla q_0, Q_0 \mathbf{u})_T - \langle q_0 - q_b, Q_0 \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla q_0, \mathbf{u})_T - \langle q_0 - q_b, Q_0 \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T}. \end{aligned}$$

Summing over all $T \in \mathcal{T}_h$ yields

$$(5.7) \quad \begin{aligned} b(Q_h \mathbf{u}, q) &= (Q_0 \mathbf{u}, \tilde{\nabla}_w q) \\ &= (\mathbf{u}, \nabla q_0) - \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, Q_0 \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T}. \end{aligned}$$

Next, using \mathbf{v}_0 in $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h$ to test (1.1), we have

$$(-\Delta \mathbf{u}, \mathbf{v}_0) + (\mathbf{v}_0, \nabla p) = (\mathbf{f}, \mathbf{v}_0).$$

Integrating by parts, we obtain

$$(5.8) \quad (\nabla \mathbf{u}, \nabla \mathbf{v}_0) + (\mathbf{v}_0, \nabla p) = (\mathbf{f}, \mathbf{v}_0) + \sum_{T \in \mathcal{T}_h} \langle \nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v}_0 - \mathbf{v}_b \rangle_{\partial T},$$

where we have used the fact that $\sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_b, \nabla \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T} = 0$. Using q_0 in $q = \{q_0, q_b\} \in W_h$ to test (1.2), we arrive at

$$(\nabla \cdot \mathbf{u}, q_0) = 0.$$

Using the fact that $\sum_{T \in \mathcal{T}_h} \langle \mathbf{u} \cdot \mathbf{n}, q_b \rangle_{\partial T} = 0$ one has

$$(5.9) \quad \begin{aligned} 0 &= (\nabla \cdot \mathbf{u}, q_0) \\ &= -(\mathbf{u}, \nabla q_0) + \sum_{T \in \mathcal{T}_h} \langle \mathbf{u} \cdot \mathbf{n}, q_0 \rangle_{\partial T} \\ &= -(\mathbf{u}, \nabla q_0) + \sum_{T \in \mathcal{T}_h} \langle \mathbf{u} \cdot \mathbf{n}, q_0 - q_b \rangle_{\partial T}. \end{aligned}$$

Finally combining the equations (5.5) and (5.6) with (5.8) yields

$$(5.10) \quad \begin{aligned} a(Q_h \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \tilde{Q}_h p) &= (\nabla_w Q_h \mathbf{u}, \nabla_w \mathbf{v}) + s(Q_h \mathbf{u}, v) + (\mathbf{v}_0, \tilde{\nabla}_w \tilde{Q}_h p) \\ &= (\mathbf{f}, \mathbf{v}_0) + \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_0 - \mathbf{v}_b, (\nabla \mathbf{u} - \mathbf{Q}_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \\ &\quad + s(Q_h \mathbf{u}, \mathbf{v}). \end{aligned}$$

Substituting it into (4.11), then we would have

$$a(\mathbf{e}_h, \mathbf{v}) + b(\mathbf{v}, \varepsilon_h) = s(Q_h \mathbf{u}, \mathbf{v}) + l_1(\mathbf{u}, \mathbf{v}).$$

Combining the equations (5.7) with (5.9) we arrive at

$$(5.11) \quad \begin{aligned} b(Q_h \mathbf{u}, q) - c(\tilde{Q}_h p, q) &= (\nabla q_0, \mathbf{u}) - \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, Q_0 \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T} - c(\tilde{Q}_h p, q) \\ &= \sum_{T \in \mathcal{T}_h} \langle \mathbf{u} \cdot \mathbf{n}, q_0 - q_b \rangle_{\partial T} - \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, Q_0 \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T} \\ &\quad - c(\tilde{Q}_h p, q) \\ &= \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, (\mathbf{u} - Q_0 \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} - c(\tilde{Q}_h p, q). \end{aligned}$$

Substituting (5.11) into (4.12) yields the following error equation

$$b(\mathbf{e}_h, q) - c(\varepsilon_h, q) = l_2(\mathbf{u}, q) - c(\tilde{Q}_h p, q),$$

for all $q \in W_h$, which completes the proof of (5.4). \square

6. Error Estimates. In this section we shall present the error estimates between the exact solution of (1.1)-(1.3) and the numerical solution of WG finite element method (4.11)-(4.12). The two norms $\|\cdot\|$ and $\|\cdot\|_0$ are essentially H^1 norm and L^2 norm on V_h and W_h respectively. In this section we always assume \mathcal{T}_h is shape regular ([18]).

THEOREM 6.1. *Let (\mathbf{u}, p) be the exact solution of (1.1)-(1.3), (\mathbf{u}_h, p_h) be the numerical solution of (4.11)-(4.12), then the following error estimates hold true*

$$(6.1) \quad \|\mathbf{e}_h\| + \|\varepsilon_h\|_* \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k),$$

$$(6.2) \quad \|\varepsilon_0\| \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k),$$

and consequently, one has

$$(6.3) \quad \|\mathbf{e}_h\| + \|\varepsilon_h\|_0 \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

Proof. Letting $\mathbf{v} = \mathbf{e}_h$ in (5.3) and $q = \varepsilon_h$ in (5.4), we would obtain

$$\begin{aligned} \|\mathbf{e}_h\|^2 + \|\varepsilon_h\|_*^2 &= s(Q_h \mathbf{u}, \mathbf{e}_h) + l_1(\mathbf{u}, \mathbf{e}_h) \\ &\quad - l_2(\mathbf{u}, \varepsilon_h) + c(\tilde{Q}_h p, \varepsilon_h). \end{aligned}$$

Then from (8.5)-(8.8) we arrive at

$$\|\mathbf{e}_h\|^2 + \|\varepsilon_h\|_*^2 \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)(\|\mathbf{e}_h\| + \|\varepsilon_h\|_*),$$

from which we would have

$$\|\mathbf{e}_h\| + \|\varepsilon_h\|_* \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

For any given $\rho \in W_h \subset L_0^2(\Omega)$, it follows from [19, 6, 7, 5, 10, 11] that there is a $\tilde{\mathbf{v}} \in [H_0^1(\Omega)]^d$ such that

$$(6.4) \quad \frac{(\nabla \cdot \tilde{\mathbf{v}}, \rho)}{\|\tilde{\mathbf{v}}\|_1} \geq C\|\rho\|,$$

where C is a positive constant which is dependent only on Ω . Let $\mathbf{v} = Q_h \tilde{\mathbf{v}} \in V_h$, we claim that the following inequality holds true

$$(6.5) \quad \|\mathbf{v}\| \leq C\|\tilde{\mathbf{v}}\|_1,$$

where C is a constant.

From (4.3), we have

$$\sum_{T \in \mathcal{T}_h} \|\nabla_w \mathbf{v}\|_T^2 = \sum_{T \in \mathcal{T}_h} \|\nabla_w(Q_h \tilde{\mathbf{v}})\|_T^2 = \sum_{T \in \mathcal{T}_h} \|\mathbf{Q}_h(\nabla \tilde{\mathbf{v}})\|_T^2 \leq \|\nabla \tilde{\mathbf{v}}\|^2.$$

It follows from the definition of Q_b , (8.1), and (8.4) that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 &= \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \tilde{\mathbf{v}} - Q_b \tilde{\mathbf{v}}\|_{\partial T}^2 \\ &= \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_b(Q_0 \tilde{\mathbf{v}}) - Q_b \tilde{\mathbf{v}}\|_{\partial T}^2 \\ &\leq \sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \tilde{\mathbf{v}} - \tilde{\mathbf{v}}\|_{\partial T}^2 \\ &\leq C\|\nabla \tilde{\mathbf{v}}\|^2, \end{aligned}$$

which yields

$$\|\mathbf{v}\| \leq C\|\tilde{\mathbf{v}}\|_1.$$

From (5.7), (6.4), (8.7), and the fact that $\sum_{T \in \mathcal{T}_h} \langle p_b, \tilde{\mathbf{v}} \cdot \mathbf{n} \rangle_{\partial T} = 0$ on $\partial\Omega$, we would obtain

$$\begin{aligned}
(6.6) \quad |b(\mathbf{v}, p)| &= |b(Q_0 \tilde{\mathbf{v}}, p)| \\
&= \left| (\nabla p_0, \tilde{\mathbf{v}}) - \sum_{T \in \mathcal{T}_h} \langle p_0 - p_b, Q_0 \tilde{\mathbf{v}} \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&= \left| -(p_0, \nabla \cdot \tilde{\mathbf{v}}) - \sum_{T \in \mathcal{T}_h} \langle p_0 - p_b, (Q_0 \tilde{\mathbf{v}} - \tilde{\mathbf{v}}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\geq |(p_0, \nabla \cdot \tilde{\mathbf{v}})| - \left| \sum_{T \in \mathcal{T}_h} \langle p_0 - p_b, (Q_0 \tilde{\mathbf{v}} - \tilde{\mathbf{v}}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\geq \|\tilde{\mathbf{v}}\|_1 (C_1 \|p_0\| - C_2 \|p\|_*).
\end{aligned}$$

Using (5.3), (6.1), (8.5) and (8.6), we obtain

$$\begin{aligned}
(6.7) \quad |b(\mathbf{v}, \varepsilon_h)| &= |s(Q_h \mathbf{u}, \mathbf{v}) + l_1(\mathbf{u}, \mathbf{v}) - a(\mathbf{e}_h, \mathbf{v})| \\
&\leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\| + \|\mathbf{e}_h\| \|\mathbf{v}\| \\
&\leq Ch^k (\|\mathbf{u}\|_{k+1} + \|p\|_k) \|\mathbf{v}\|.
\end{aligned}$$

Let \mathbf{v} be such that (6.6) is true, it follows from (6.7) that

$$Ch^k (\|\mathbf{u}\|_{k+1} + \|p\|_k) \geq \frac{|b(\mathbf{v}, \varepsilon_h)|}{\|\mathbf{v}\|} \geq C \frac{|b(\mathbf{v}, \varepsilon_h)|}{\|\tilde{\mathbf{v}}\|_1} \geq C_1 \|\varepsilon_0\| - C_2 \|\varepsilon_h\|_*.$$

Then (6.1) implies that

$$\|\varepsilon_0\| \leq Ch^k (\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

From what we have demonstrated, one has

$$\|\mathbf{e}_h\| + \|\varepsilon_h\|_0 \leq Ch^k (\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

□

We shall use the dual technique to derive the L^2 error. Assume this problem has the $[H^2(\Omega)]^d \times H^1(\Omega)$ -regularity, then the solution $(\psi; \xi) \in [H^2(\Omega)]^d \times H^1(\Omega)$ of the following equations

$$(6.8) \quad -\Delta \psi + \nabla \xi = \mathbf{e}_0, \quad \text{in } \Omega,$$

$$(6.9) \quad \nabla \cdot \psi = 0, \quad \text{in } \Omega,$$

$$(6.10) \quad \psi = 0, \quad \text{on } \partial\Omega$$

satisfies the following property

$$(6.11) \quad \|\psi\|_2 + \|\xi\|_1 \leq C \|\mathbf{e}_0\|.$$

THEOREM 6.2. Let $(\mathbf{u}; p) \in [H^1(\Omega) \cap H^{k+1}(\Omega)]^d \times (L_0^2 \cap H^k(\Omega))$ be the exact solution of (1.1)-(1.3), $(\mathbf{u}_h; p_h) \in V_h \times W_h$ be the numerical solution of (4.11)-(4.12), $\mathbf{e}_0 = Q_0 \mathbf{u} - \mathbf{u}_0$ in $\mathbf{e} = \{\mathbf{e}_0, \mathbf{e}_h\}$ then the following error estimate holds true

$$(6.12) \quad \|\mathbf{e}_0\| \leq Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

Proof. Since $(\psi; \xi)$ is the solution of (6.8)-(6.10), letting $\mathbf{u} = \psi$, $\mathbf{v} = \mathbf{e}_h$, $p = \xi$ and $f = \mathbf{e}_0$ in (5.10) gives

$$(6.13) \quad \|\mathbf{e}_0\|^2 = a(Q_h \psi, \mathbf{e}_h) + b(\mathbf{e}_h, \tilde{Q}_h \xi) - l_1(\psi, \mathbf{e}_h) - s(Q_h \psi, \mathbf{e}_h).$$

Letting $q = \tilde{Q}_h \xi$ in (5.4), we obtain

$$(6.14) \quad b(\mathbf{e}_h, \tilde{Q}_h \xi) = c(\varepsilon_h, \tilde{Q}_h \xi) + l_2(\mathbf{u}, \tilde{Q}_h \xi) - c(\tilde{Q}_h p, \tilde{Q}_h \xi).$$

From (5.7) and (6.9), we arrive at

$$(6.15) \quad \begin{aligned} b(Q_h \psi, \varepsilon_h) &= (\nabla \varepsilon_0, \psi) - \sum_{T \in \mathcal{T}_h} \langle \varepsilon_0 - \varepsilon_b, Q_0 \psi \cdot \mathbf{n} \rangle_{\partial T} \\ &= -(\varepsilon_0, \nabla \cdot \psi) + \sum_{T \in \mathcal{T}_h} \langle \varepsilon_0, \psi \cdot \mathbf{n} \rangle_{\partial T} - \sum_{T \in \mathcal{T}_h} \langle \varepsilon_0 - \varepsilon_b, Q_0 \psi \cdot \mathbf{n} \rangle_{\partial T} \\ &= -(\varepsilon_0, \nabla \cdot \psi) + \sum_{T \in \mathcal{T}_h} \langle \varepsilon_0 - \varepsilon_b, (\psi - Q_0 \psi) \cdot \mathbf{n} \rangle_{\partial T} \\ &= l_2(\psi, \varepsilon_h), \end{aligned}$$

where we have used the fact that $\sum_{T \in \mathcal{T}_h} \langle \varepsilon_b, \psi \cdot \mathbf{n} \rangle_{\partial T} = 0$.

Taking $\mathbf{v} = Q_h \psi$ in (5.3), combined with (6.13)-(6.15) one has

$$(6.16) \quad \begin{aligned} \|\mathbf{e}_0\|^2 &= s(Q_h \mathbf{u}, Q_h \psi) + l_1(\mathbf{u}, Q_h \psi) \\ &\quad - l_2(\psi, \varepsilon_h) + c(\tilde{Q}_h \xi, \varepsilon_h) + l_2(\mathbf{u}, \tilde{Q}_h \xi) \\ &\quad - c(\tilde{Q}_h p, \tilde{Q}_h \xi) - l_1(\psi, \mathbf{e}_h) - s(Q_h \psi, \mathbf{e}_h). \end{aligned}$$

It follows from (8.5)-(8.6) that

$$(6.17) \quad |l_1(\psi, \mathbf{e}_h) + s(Q_h \psi, \mathbf{e}_h)| \leq Ch(\|\psi\|_2 + \|\xi\|_1) \|\mathbf{e}_h\|.$$

Using (8.7)-(8.8), we would obtain

$$(6.18) \quad |l_2(\psi, \varepsilon_h) - c(\tilde{Q}_h \xi, \varepsilon_h)| \leq Ch(\|\psi\|_2 + \|\xi\|_1) \|\varepsilon_h\|_*.$$

From the definition of Q_b , (8.1), (8.2), and (8.4), we arrive at

$$\begin{aligned}
(6.19) \quad |l_1(\mathbf{u}, Q_h \psi)| &= \left| \sum_{T \in \mathcal{T}_h} \langle Q_0 \psi - Q_b \psi, (\nabla \mathbf{u} - Q_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} \langle Q_b(Q_0 \psi) - Q_b \psi, (\nabla \mathbf{u} - Q_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} \langle Q_b(Q_0 \psi - \psi), (\nabla \mathbf{u} - Q_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\leq \|Q_0 \psi - \psi\|_{\partial T} \|\nabla \mathbf{u} - Q_h \nabla \mathbf{u}\|_{\partial T} \\
&\leq Ch^{k+1} \|\psi\|_2 \|\mathbf{u}\|_{k+1}.
\end{aligned}$$

It follows from the definition of Q_b , (8.4), and (8.1) that

$$\begin{aligned}
(6.20) \quad |s(Q_h \mathbf{u}, Q_h \psi)| &= \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 \mathbf{u} - Q_b \mathbf{u}, Q_0 \psi - Q_b \psi \rangle_{\partial T} \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b(Q_0 \mathbf{u} - \mathbf{u}), Q_b(Q_0 \psi - \psi) \rangle_{\partial T} \right| \\
&\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \mathbf{u} - \mathbf{u}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \psi - \psi\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^{k+1} \|\mathbf{u}\|_{k+1} \|\psi\|_2.
\end{aligned}$$

The definition of Q_b together with (8.1), (8.3), and (8.4) yields

$$\begin{aligned}
(6.21) \quad |l_2(\mathbf{u}, \tilde{Q}_h \xi)| &= \left| \sum_{T \in \mathcal{T}_h} \langle \tilde{Q}_0 \xi - \tilde{Q}_b \xi, (\mathbf{u} - Q_0 \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} \langle \tilde{Q}_b(\tilde{Q}_0 \xi) - \tilde{Q}_b \xi, (\mathbf{u} - Q_0 \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} \langle \tilde{Q}_b(\tilde{Q}_0 \xi - \xi), (\mathbf{u} - Q_0 \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\leq \left(\sum_{T \in \mathcal{T}_h} \|\tilde{Q}_0 \xi - \xi\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} \|\mathbf{u} - Q_0 \mathbf{u}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^{k+1} \|\xi\|_1 \|\mathbf{u}\|_{k+1}.
\end{aligned}$$

From the definition of Q_b , (8.3), and (8.4), we obtain

$$\begin{aligned}
|c(\tilde{Q}_h p, \tilde{Q}_h \xi)| &= \left| \sum_{T \in \mathcal{T}_h} h_T \langle \tilde{Q}_0 p - \tilde{Q}_b p, \tilde{Q}_0 \xi - \tilde{Q}_b \xi \rangle_{\partial T} \right| \\
(6.22) \quad &= \left| \sum_{T \in \mathcal{T}_h} h_T \langle \tilde{Q}_b(\tilde{Q}_0 p - p), \tilde{Q}_b(\tilde{Q}_0 \xi - \xi) \rangle_{\partial T} \right| \\
&\leq \left(\sum_{T \in \mathcal{T}_h} h_T \|\tilde{Q}_0 p - p\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T \|\tilde{Q}_0 \xi - \xi\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^{k+1} \|p\|_k \|\xi\|_1.
\end{aligned}$$

From (6.16)-(6.22), one has

$$\|\mathbf{e}_0\|^2 \leq Ch^{k+1} (\|\psi\|_2 + \|\xi\|_1) (\|\mathbf{u}\|_{k+1} + \|p\|_k) + Ch (\|\psi\|_2 + \|\xi\|_1) (\|\mathbf{e}_h\| + \|\varepsilon_h\|_*),$$

it follows from (6.11) that

$$\|\mathbf{e}_0\| \leq Ch^{k+1} (\|\mathbf{u}\|_{k+1} + \|p\|_k) + Ch (\|\mathbf{e}_h\| + \|\varepsilon_h\|_*),$$

together with (6.1), we would have

$$\|\mathbf{e}_0\| \leq Ch^{k+1} (\|\mathbf{u}\|_{k+1} + \|p\|_k),$$

which completes the proof of the theorem. \square

7. Numerical Experiments. The goal of this section is to report some numerical results for the weak Galerkin finite element method proposed and analyzed in previous sections.

Let $(\mathbf{u}; p)$ be the exact solution of (1.1)-(1.3) and $(\mathbf{u}_h; p_h)$ be the numerical solution of (4.11)-(4.12). Denote $\mathbf{e}_h = Q_h \mathbf{u} - \mathbf{u}_h$ and $\varepsilon_h = \tilde{Q}_h p - p_h$. The error for the weak Galerkin solution is measured in four norms defined as follows:

$$\begin{aligned}
\|\mathbf{e}_h\|^2 &= \sum_{T \in \mathcal{T}_h} \left(\int_T |\nabla_w \mathbf{e}_h|^2 dT + h_T^{-1} \int_{\partial T} (\mathbf{e}_0 - \mathbf{e}_b)^2 ds \right), \\
\|\mathbf{e}_h\|^2 &= \sum_{T \in \mathcal{T}_h} \int_T |\mathbf{e}_h|^2 dT, \\
\|\varepsilon_h\|_0^2 &= \sum_{T \in \mathcal{T}_h} \left(\int_T |\varepsilon_0|^2 dT + h_T \int_{\partial T} |\varepsilon_0 - \varepsilon_b|^2 ds \right), \\
\|\varepsilon_h\|^2 &= \sum_{T \in \mathcal{T}_h} \int_T |\varepsilon_h|^2 dT.
\end{aligned}$$

Example 7.1 Consider the problem (1.1)-(1.3) in the square domain $\Omega = (0, 1)^2$. The WG finite element space $k = 1$ is employed in the numerical discretization. It has the analytic solution

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x) \sin(\pi y) \\ \cos(\pi x) \cos(\pi y) \end{pmatrix} \text{ and } p = 2 \cos(\pi x) \sin(\pi y).$$

The right hand side function \mathbf{f} in (1.1) is computed to match the exact solution. The mesh size is denoted by h .

Table 7.1 shows that the errors and convergence rates of Example 7.1 in $\|\cdot\|$ -norm and L^2 -norm for the WG-FEM solution \mathbf{u} are of order $O(h)$ and $O(h^2)$ when $k = 1$, respectively.

Table 7.2 shows that the errors and orders of Example 7.1 in $\|\cdot\|_0$ -norm and L^2 -norm for pressure when $k = 1$. The numerical results are also consistent with theory for these two cases.

Table 7.3 and 7.4 show the errors and orders of Example 7.1 for the case $k = 2$, and the convergence rates coincide with the theoretical expectation.

Table 7.1. Numerical errors and orders for \mathbf{u} of Example 7.1 with $k = 1$.

h	$\ \mathbf{e}_h\ $	order	$\ \mathbf{e}_h\ $	order
1/4	1.2347e+00		1.0681e-01	
1/8	7.5411e-01	0.7113	2.8345e-02	1.9139
1/16	4.0953e-01	0.8808	7.8149e-03	1.8588
1/32	2.0483e-01	0.9995	2.0169e-03	1.9541
1/64	1.0172e-01	1.0099	5.0860e-04	1.9876
1/128	5.0471e-02	1.0110	1.2745e-04	1.9966

Table 7.2. Numerical errors and orders for p of Example 7.1 with $k = 1$.

h	$\ \varepsilon_h\ _0$	order	$\ \varepsilon_h\ $	order
1/4	1.1642e+00		7.6948e-01	
1/8	5.1214e-01	1.1847	3.4266e-01	1.1671
1/16	2.1109e-01	1.2786	1.1063e-01	1.6311
1/32	9.3992e-02	1.1673	3.1403e-02	1.8168
1/64	4.4978e-02	1.0633	8.8252e-03	1.8312
1/128	2.2179e-02	1.0200	2.6757e-03	1.7217

Table 7.3. Numerical errors and orders for \mathbf{u} of Example 7.1 with $k = 2$.

h	$\ \mathbf{e}_h\ $	order	$\ \mathbf{e}_h\ $	order
1/4	2.3130e-01		1.8281e-02	
1/8	5.9891e-02	1.9494	2.3513e-03	2.9588
1/16	1.4832e-02	2.0136	2.9732e-04	2.9834
1/32	3.7134e-03	1.9979	3.7349e-05	2.9929
1/64	9.2987e-04	1.9977	4.6792e-06	2.9967
1/128	2.3265e-04	1.9989	5.8553e-07	2.9984

Table 7.4. Numerical errors and orders for p of Example 7.1 with $k = 2$.

h	$\ \varepsilon_h\ _0$	order	$\ \varepsilon_h\ $	order
1/4	1.9753e-01		4.3517e-02	
1/8	5.0278e-02	1.9741	8.5422e-03	2.3489
1/16	1.2600e-02	1.9965	1.8980e-03	2.1702
1/32	3.1485e-03	2.0007	4.5063e-04	2.0744
1/64	7.8661e-04	2.0010	1.1009e-04	2.0333
1/128	1.9657e-04	2.0006	2.7227e-05	2.0155

Example 7.2 Consider the problem (1.1)-(1.3) in the square domain $\Omega = (0, 1)^2$. The WG finite element space $k = 1$ is employed in the numerical discretization. It has the analytic solution

$$\mathbf{u} = \begin{pmatrix} 2\pi \sin^2(\pi x) \cos(\pi y) \sin(\pi y) \\ -2\pi \sin(\pi x) \cos(\pi x) \sin^2(\pi y) \end{pmatrix}$$

and

$$p = \cos(\pi x) \cos(\pi y).$$

The right hand side function \mathbf{f} in (1.1) is computed to match the exact solution. The mesh size is denoted by h .

The numerical results are presented in Tables 7.5-7.8, which confirm the theory developed in previous sections.

Table 7.5. Numerical errors and orders for \mathbf{u} of Example 7.2 with $k = 1$.

h	$\ \mathbf{e}_h\ $	order	$\ \mathbf{e}_h\ $	order
1/4	1.3024e+01		1.9402e+00	
1/8	6.2924e+00	1.0494	3.1369e-01	2.6288
1/16	3.1404e+00	1.0027	5.7291e-02	2.4530
1/32	1.5840e+00	0.9874	1.2695e-02	2.1740
1/64	7.9695e-01	0.9910	3.0804e-03	2.0431
1/128	3.9961e-01	0.9959	7.6596e-04	2.0078

Table 7.6. Numerical errors and orders for p of Example 7.2 with $k = 1$.

h	$\ \varepsilon_h\ _0$	order	$\ \varepsilon_h\ $	order
1/4	2.5875e+00		6.7394e-01	
1/8	1.1518e+00	1.1676	5.3190e-01	0.3415
1/16	5.1671e-01	1.1565	2.8686e-01	0.8908
1/32	2.2432e-01	1.2038	1.1558e-01	1.3114
1/64	9.9999e-02	1.1656	4.1483e-02	1.4783
1/128	4.6840e-02	1.0942	1.5196e-02	1.4489

Table 7.7. Numerical errors and orders for \mathbf{u} of Example 7.2 with $k = 2$.

h	$\ \mathbf{e}_h\ $	order	$\ \mathbf{e}_h\ $	order
1/4	3.1376e+00		2.4616e-01	
1/8	8.2291e-01	1.9309	3.2044e-02	2.9415
1/16	2.0280e-01	2.0207	4.0772e-03	2.9744
1/32	5.0679e-02	2.0006	5.1315e-04	2.9901
1/64	1.2698e-02	1.9968	6.4328e-05	2.9959
1/128	3.1774e-03	1.9986	8.0513e-06	2.9981

Table 7.8. Numerical errors and orders for p of Example 7.2 with $k = 2$.

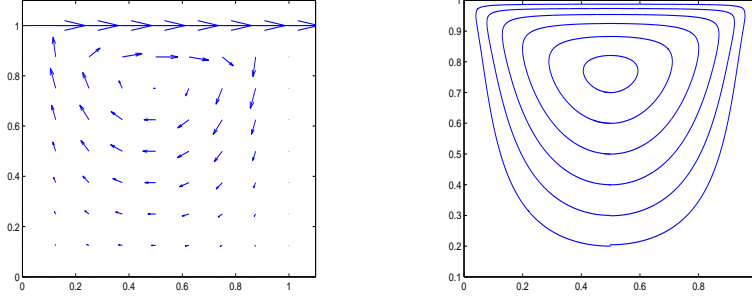


FIG. 7.1. *Left panel: The vectorgraph of velocity; Right panel: The streamlines of velocity*

h	$\ \varepsilon_h\ _0$	order	$\ \varepsilon_h\ $	order
1/4	1.3811e+00		4.8957e-01	
1/8	3.5346e-01	1.9662	8.5399e-02	2.5192
1/16	8.7131e-02	2.0203	1.5911e-02	2.4242
1/32	2.1567e-02	2.0144	3.4240e-03	2.2163
1/64	5.3636e-03	2.0075	7.9926e-04	2.0990
1/128	1.3374e-03	2.0038	1.9366e-04	2.0452

Example 7.3 Consider the following lid-driven cavity problem. in the square domain $\Omega = (0,1)^2$. This is a benchmark testcase for Stokes flow, which has been tested in [9, 13, 22, 23]. A delicate analysis of solution regularity is presented in [3].

In this example, a uniform mesh with step $h = 1/32$ and polynomial degree $k = 2$ are employed. The source term in (1.1) is $\mathbf{f} = 0$ and the Dirichlet boundary condition is given as

$$\mathbf{u} = \begin{cases} (1, 0)^T, & \text{if } x = 1, y \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

The exact solution of lid-driven cavity problem is unknown, which has singularity at point $(1, 0)$ and $(1, 1)$.

The vectograph and streamlines of the velocity field are presented in Fig 7.1 and Fig7.2. The shape of streamlines is similar to the result obtained by IFISS[8].

8. Appendix. In this section, we will give some important inequalities.

LEMMA 8.1. ([18, 21]) *Assume that \mathcal{T}_h , the finite element partition of Ω , is shape regular. Let $\mathbf{u} \in [H^{r+1}(\Omega)]^d$ and $p \in H^r(\Omega)$ with $1 \leq r \leq k$. Then for any $0 \leq s \leq 1$ we have*

$$(8.1) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\mathbf{u} - Q_0 \mathbf{u}\|_{T,s}^2 \leq Ch^{2(r+1)} \|\mathbf{u}\|_{r+1}^2,$$

$$(8.2) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\nabla \mathbf{u} - \mathbf{Q}(\nabla \mathbf{u})\|_{T,s}^2 \leq Ch^{2r} \|\mathbf{u}\|_{r+1}^2,$$

$$(8.3) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|p - \tilde{Q}_0 p\|_{T,s}^2 \leq Ch^{2r} \|p\|_r^2,$$

where C is a constant which is independent of the meshsize h and the functions.

LEMMA 8.2. ([21]) Let T be an element of the finite element partition \mathcal{T}_h , e is an edge or face which is part of ∂T . For any function $g \in H^1(T)$, the following trace inequality holds true

$$(8.4) \quad \|g\|_e^2 \leq C(h_T^{-1}\|g\|_T^2 + h_T\|\nabla g\|_T^2).$$

LEMMA 8.3. Let $\mathbf{u} \in [H_0^{k+1}(\Omega)]^d$, $p \in H^k(\Omega)$, $\mathbf{v} \in V_h$ and $q \in W_h$. The following estimates hold true

$$(8.5) \quad |s(Q_h \mathbf{u}, \mathbf{v})| \leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\|,$$

$$(8.6) \quad |l_1(\mathbf{u}, \mathbf{v})| \leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\|,$$

$$(8.7) \quad |l_2(\mathbf{u}, q)| \leq Ch^k \|\mathbf{u}\|_{k+1} \|q\|_*,$$

$$(8.8) \quad |c(\tilde{Q}_h p, q)| \leq Ch^k \|p\|_k \|q\|_*.$$

Proof. It follows from the definition of Q_b , (8.1), and (8.4) that

$$\begin{aligned} |s(Q_h \mathbf{u}, \mathbf{v})| &= \left| \sum_{T \in \mathcal{T}_h} h_T^{-1} (Q_0 \mathbf{u} - Q_b \mathbf{u}, \mathbf{v}_0 - \mathbf{v}_b)_{\partial T} \right| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|Q_0 \mathbf{u} - \mathbf{u}\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\|. \end{aligned}$$

From (8.2) and (8.4), we would have

$$\begin{aligned} |l_1(\mathbf{u}, \mathbf{v})| &= \left| \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_0 - \mathbf{v}_b, (\nabla \mathbf{u} - \mathbf{Q}_h \nabla \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T \|\nabla \mathbf{u} - \mathbf{Q}_h \nabla \mathbf{u}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\leq Ch^k \|\mathbf{u}\|_{k+1} \|\mathbf{v}\|. \end{aligned}$$

Using (8.1) and (8.4), we would arrive at

$$\begin{aligned} |l_2(\mathbf{u}, q)| &= \left| \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, (\mathbf{u} - Q_0 \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\ &\leq \left(\sum_{T \in \mathcal{T}_h} h_T \|q_0 - q_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T^{-1} \|\mathbf{u} - Q_0 \mathbf{u}\|_{\partial T}^2 \right)^{\frac{1}{2}} \\ &\leq Ch^k \|\mathbf{u}\|_{k+1} \|q\|_*. \end{aligned}$$

From the definition of Q_b and (8.3), we obtain

$$\begin{aligned}
|c(\tilde{Q}_h p, q)| &= \left| \sum_{T \in \mathcal{T}_h} h_T \langle \tilde{Q}_0 p - \tilde{Q}_b p, q_0 - q_b \rangle_{\partial T} \right| \\
&= \left| \sum_{T \in \mathcal{T}_h} h_T \langle \tilde{Q}_0 p - p, q_0 - q_b \rangle_{\partial T} \right| \\
&\leq \left(\sum_{T \in \mathcal{T}_h} h_T \|\tilde{Q}_0 p - p\|_{\partial T}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}_h} h_T \|q_0 - q_b\|_{\partial T}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^k \|p\|_k \|q\|_*.
\end{aligned}$$

Now we have proved all the estimates in this lemma. \square

REFERENCES

- [1] I. Babuska, The finite method with Lagrangian multiplier, Numer.cvMath. 20(1973).
- [2] S. Brenner, R. Scott, "The mathematical theory of finite element methods", Springer, 3rd ed., 2008.
- [3] Z.Cai and Y.Wang, "An error estimate for two-dimensional Stokes driven cavity flow", Math. Comp., 78(2009), pp.771-87.
- [4] B. Cockburn, J. Gopalakrishnan, "Incompressible finite elements via hybridization. I. The Stokes system in two space dimensions", SIAM J. Numer. Anal. 43(2005), pp.1627-1650.
- [5] M. Crouzeix, P. A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, RAIRO Anal. Numer. 7 (1973) 33-76.
- [6] S. Brenner, R. Scott, Mathematical theory of finite element methods, Springer, 2002.
- [7] F. Brezzi, M. Fortin, Mixed and hybrid finite elements, Springer-Verlag, New York, 1991.
- [8] H. Elman, A. Ramage, D. Silvester, Algorithm 866: IFISS, a Matlab toolbox for modelling incompressible flow, ACM Trans. Math. Softw. 33 (2007) 2-14.
- [9] H. Elman, D. Silvester, A. Wathen, "Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics", Oxford University Press, Oxford (2005).
- [10] V. Girault, P. A. Raviart, Finite element methods for the Navier-Stokes equations theory and algorithms, Springer-Verlag, Berlin, 1986.
- [11] M. D. Gunzburger, Finite element methods for Viscous Incompressible Flows, A Guide to theory, practice and Algorithms, Academic, San Diego, 1989.
- [12] O. Karakashian and T. Katsaounis, "Numerical simulation of incompressible fluid flow using locally solenoidal elements", Comput. Math. Appl., 51(2006), pp.1551-570.
- [13] J. Liu, "Penalty-factor-free discontinuous Galerkin methods for 2-dim Stokes problems", SIAM J. Numer. Anal., 49(2011), pp.2165-2181.
- [14] L. Mu, J. Wang, X. Ye, S.Zhang A weak Galerkin finite element method for the Maxwell equations, J. Sci. Comput. 65(1) (2015) 363-386.
- [15] L. Mu, J. Wang, X. Ye, Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes, Numerical Methods for PDEs. 30 (2014) 1003-1029.
- [16] L. Mu, J. Wang, X. Ye, S. Zhang, A C^0 -weak Galerkin finite element method for the biharmonic equation, J. Sci. Comput. 59(2)(2014) 473-495.
- [17] L. Mu, J. Wang, X. Ye, A stable numerical algorithm for the Brinkman equations by weak Galerkin finite element methods, J. of Comput. Phys. 273 (2014) 327-342.
- [18] L. Mu, J. Wang, X. Ye, Weak Galerkin finite element methodes on polytopal meshes, Int. J. Numer. Anal. Model. 12(1) (2015) 31-53.
- [19] J. Wang, X. Ye, A weak Galerkin finite element method for the Stokes equations, arXiv:1302.2707v1.
- [20] J. Wang, X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comp. Appl. Math. 241 (2013) 103-115.
- [21] J. Wang, X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, Math. Comp. 83(289) (2014) 2101-2126.
- [22] J. Wang, Y. Wang, X. Ye, "A robust numerical method for Stokes equations based on

- divergence-free $H(\text{div})$ finite element methods", *SIAM J. Sci. Comput.*, 31(2009), pp.2784–2802.
- [23] M. Yang, J. Liu, Y. Lin, "Pressure recovery for the weakly over-penalized discontinuous Galerkin methods for the Stokes problem", *J. Sci. Comput.*, 63(2015), pp.699–715.
 - [24] Q. Zhai, R. Zhang, L. Mu, A New Weak Galerkin Finite Element Scheme for the Brinkman Model, *Commun. Comp. Phys.* accepted.
 - [25] Q. Zhai, R. Zhang, X. Wang, A hybridized weak Galerkin finite element scheme for the Stokes equations, *Sci China Math.* 58 (2015) 2455-2472.
 - [26] R. Zhang, Q. Zhai, A new weak Galerkin finite element scheme for the biharmonic equations by using polynomials of reduced order, *J. Sci. Comput.* 64(2) (2015) 559-585.